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SHOULD PLATO'S LINE BE DIVIDED IN THE MEAN AND EXTREME RATIO?

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Abstract

Des Jardins (1976) and Dreher (1990) have suggested that Plato's Line should be thought of as divided in the mean and extreme ('golden') ratio. I examine their arguments, as well as other reasons that could be brought up in support of the 'golden division' of the Line, and show that all of them are wanting.

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1. Introduction.

To represent the realms of the visible and of the intelligible, Plato invokes, at *Republic* 506d6–511e5 and 534a, an image of a line divided into two unequal sections. Each section is in turn divided into two subsections in the same ratio (see Figure 1). The subsections of the visible comprise (A) images (shadows, reflections in water and the like) and (B) the objects imaged (animals, plants, man-made things). The comparative 'clearness and obscurity' of these subsections, or their relative status with respect to reality and truth, are represented by a ratio equal to that of $A+B$ to the two sections of the intelligible realm $C+D$. The same ratio relates the two subsections of the intelligible.[1] One of its subsections (C) can only be studied by the soul through the use of images. Such is the method practiced by 'students of geometry and reckoning' (510c)[2] who postulate their objects hypothetically, regard them as unproblematic and known to everybody and use them for drawing out particular consequences. Though they use diagrams and other visible representations (say, of a square or of a diagonal), they are actually thinking of those things of which these are a likeness (namely, of the square as such and the diagonal as such). The soul involved in this kind of investigation is unable to extricate itself from and rise above its assumptions. It uses as images the things that are themselves imaged in the lower realm of the visible. The other subsection of the intelligibles (D) is studied by the power of dialectic inherent in reason. It regards its assumptions and hypotheses not as absolute beginnings but as 'footings' and 'springboards' for rising to 'that which requires no assumption'. Having attended to this nonhypothetical principle, the reason can then proceed downward to the conclusion, always remaining in the realm of ideas and making no use of sensible objects. Corresponding to the four sections of the intelligible and the visible (in decreasing order of the degree in which their objects partake of truth and reality) are the four 'affections of the soul' or mental states (in decreasing order of the degree in which they participate in clearness and precision): intellection or reason (*noēsis*), understanding (*dianoia*), belief (*pistis*), and 'picture thinking' or conjecture (*eikasia*).[3]

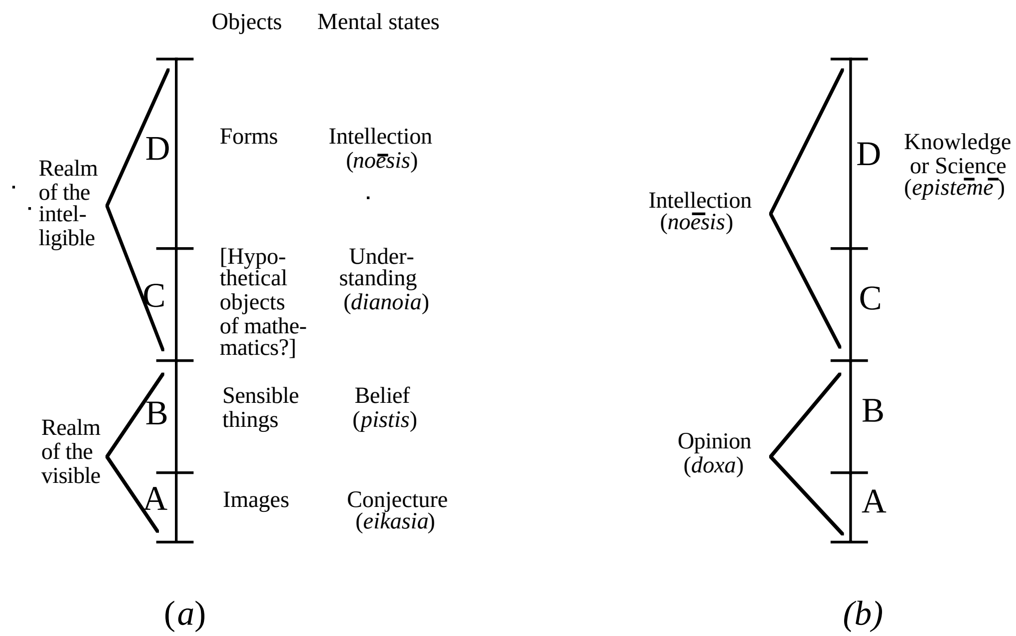

Figure 1. Plato's Divided Line. (a) (506d6–511e5) $\frac{B}{A} = \frac{D}{C} = \frac{C+D}{A+B}$. As a consequence, $B = C$. (b) (534a) B and C interchanged: $\frac{C+D}{A+B} = \frac{D}{B} = \frac{C}{A}$.

At 534a, Plato substitutes *epistēmē* for *noēsis* (section D) while reserving *noēsis* for the whole region of the intelligible (see Figure 1b). Interestingly, he then makes Socrates explicitly *interchange* B and C: 'As intellection $[C+D]$ is to opinion $[A+B]$, so is science $[D]$ to belief $[B]$, and understanding $[C]$ to image thinking $[A]$', that is, $\frac{C+D}{A+B} = \frac{D}{B} = \frac{C}{A}$.[4]

Plato's description of the Divided Line is notoriously incomplete. What is the length of the Line? In what particular ratio should it be divided and subdivided? Should one draw it vertically or horizontally?

One can easily argue that the length of the Line really does not matter. What matters are the ratios of its sections. As to the orientation, it may be conjectured to be vertical rather than horizontal, for it portrays a certain gradation of objects and mental states, at least some of which are related to each other as the image to the imaged,[5] that is, as 'inferior' (in some sense) to 'superior'. Thus, according to Raven (1953, 24), the Line should definitely be drawn vertically, for it 'presents a vertical scale of reality'. However, there is nothing in the relevant passages of the *Republic* that would explicitly endorse this view. Moreover, despite the fact that almost all scholars[6] quite naturally assign greater lengths to B, D, and $C+D$ in comparison with A, C, and $A+B$ respectively, relative magnitude is nowhere in Plato correlated with relative superiority of mental states.

Some authors (Rose 1964) go so far as to question the quadripartite line imagery on the whole and to suggest instead a two-dimensional diagram for representing Plato's division according to objects and mental states. Whether or not there is room in Plato's text for such extreme interpretations, there is certainly enough ambiguity to accommodate rather divergent readings.[7]

With regard to the particular ratio in which the Line must be divided, this indeterminacy boils down to the two possible kinds of division—'rational' and 'irrational'.[8] In the first case, the lengths of the four sections—A, B, C, D—can be represented as $\{a,\ ar,\ ar,\ ar^2\}$, where the basic ratio r is rational. In the second case, r is irrational. Since the most common (though by no means unique) class of irrationals with which pre-Euclidean mathematicians dealt explicitly is that formed by square roots of non-square numbers, one can further classify the 'irrational' way of dividing the Line into the 'simple-irrational' subclass: $r=\sqrt{m}$ (m is neither square nor ratio of squares), and the 'composite-irrational' one: $r=\sqrt{m}+\sqrt{n}$, (where not both m and n are squares or ratios of squares). Typical examples of the 'rational' division of the Line would be $\{1,\ 5,\ 5,\ 25\}$, $(a=1,\ r=5)$ and $\{\sqrt{3},\ 2\sqrt{3},\ 2\sqrt{3},\ 4\sqrt{3}\}$, $(a=\sqrt{3},\ r=2)$.[9] Two cognate examples of the 'simple-irrational' divisions are $\{1,\ \sqrt{5},\ \sqrt{5},\ 5\}$, $(a=1,\ r=\sqrt{5})$, and $\{\sqrt{3},\ \sqrt{6},\ \sqrt{6},\ \sqrt{12}\}$, $(a=\sqrt{3},\ r=\sqrt{2})$. One example of the ‘composite-irrational’ division is $\{\sqrt{11}-\sqrt{3},\ 2\sqrt{2},\ 2\sqrt{2},\ \sqrt{11}+\sqrt{3}\}$, ($a=\sqrt{11}-\sqrt{3},\ r=\sqrt{\frac{11}{8}}+\sqrt{\frac{3}{8}}$).

A very important instance of the ‘composite-irrational’ division is $r=\phi$, where $\phi=\frac{\sqrt{5}+1}{2}\approx 1.61803$ is the *golden ratio* (‘the mean and extreme ratio’). Some commentators have recently suggested that the Line should be thought of as divided in the mean and extreme ratio (Des Jardins 1976, Dreher 1990).[10] The arguments offered in support of this suggestion are rather diverse, and further arguments are available. This essay attempts to examine such arguments. It will turn out that none of them is strong enough to withstand counterevidence and to establish the golden division of the Line with certainty. Even though this problem (as many other riddles of the Platonic corpus) admits of no definite solution, the arguments for and against the golden division of the Line are interesting by themselves and possibly can throw light on other controversial issues.

2. Arguments for Irrationality of r

As a first step, one might attempt to demonstrate that the division of the Line cannot be rational. Des Jardins (1976, 485-486) suggested two arguments to this effect. One addresses the question why Plato chose a geometrical, rather than an arithmetic, representation of the ratios between the four kinds of objects/mental states. The other concerns his use of a linear, rather than, say, a two-dimensional, structure for such a representation.[11] Des Jardins seems to argue that the best explanation of both facts is that Plato wanted the basic ratio of his division to be irrational.

Indeed, a rational ratio (*logos)* can be expressed by numbers, arithmetically. One need not appeal to geometrical imagery for this.[12] The relation of incommensurables, however, is ‘without ratio’ (*alogoi*). A proportion of pairs of incommensurable magnitudes could (at Plato’s time) only be established geometrically. Plato, therefore, would have a good reason to employ geometrical imagery if he really had in mind the irrational division of the Line. 'It is only if the ratio of kinds indicates an incommensurable division that one needs lines instead of numbers' (Des Jardins 1976, 485).

This argument is not persuasive. To be sure, the relation of incommensurables could be represented geometrically. As we shall see, however, certain irrational magnitudes, including ϕ, could also be represented by *converging series* of rational ratios (see, *e.g.*, Dreher 1990). There was also a means of dealing with equalities of irrational ratios in terms of numbers, possibly available to Plato, that was based on Eudoxus' theory of proportions later summarized in Book V of Euclid's *Elements*.[13] The use of the geometrical image thus is not necessitated by the intention to express an irrational magnitude and, hence, cannot signal such an intention with any certainty.

Des Jardins's second argument is based on the fact that only prime numbers are truly 'linear'. All composite numbers can just as well be symbolized by rectangles or squares. Since Plato does not even consider 'non-linear' alternatives for his division, the Line stands for a prime number. If so, it cannot be *rationally* divided so that $\frac{B}{A}=\frac{D}{C}=\frac{C+D}{A+B}$. As Des Jardins puts it, 'no line consisting of a prime number of units allows proportional division and subdivision whose parts are commensurable by whole numbers' (Des Jardins 1976, 485-486). Put in my notation, his conclusion is that if the length of the Line, $l=a+ar+ar+ar^2=a(1+2r+r^2)=a(1+r)^2$, is a prime number, then $r=\sqrt{\frac{l}{a}}-1$ cannot be rational.

This conclusion, however, is in error.[14] Take $l=5$, a prime number, and $a=\frac{1}{5}$. This gives $r=4$, clearly a rational number. Contrary to Des Jardins, there appears to be no way of explaining Plato's choice of the line imagery on mathematical grounds alone. If an explanation is there to be found, it probably should be sought in the actual uses to which the imagery is put, including parallels with the 'linear' ascent from the Cave.

3. Metaphysical Arguments

A more direct argument for $r = \phi$, put forward by Des Jardins, is based on metaphysical rather than mathematical considerations. He believes that the correspondence between the numerical ratios of sections in the Line and the 'ratios' of the kinds of objects (mental states) represented by these sections is more revealing than it might originally appear. In particular, '*eikones*, having no depth, and the thinkable as a whole are alike in kind insofar as neither has body' (1976, 491). This similarity of *eikones* or images (A) and the whole realm of the intelligible ($C + D$) can be expressed as $A = C + D$.[15]

Des Jardins's procedure of expressing the relevant similarity by equating the lengths of the corresponding sections of the Line is not intuitively convincing and relies upon his arguments for support. Visible bodies and the objects of mathematics,[16] he argues, are also similar in that, unlike *eikones* and the Forms, they both are of the *quantifiable* kind. According to the adopted procedure, one should equate the lengths of B and C, and this finds an *independent* corroboration in Plato.[17] Having demonstrated that the procedure works in one case, one can now rely on it in another case and conclude that $A = C + D$. Together with $\frac{A}{C} = \frac{C}{D}$, this gives, for the basic ratio of the Line, $r = \frac{C}{D} = \phi$.

Two main objections can be raised against Des Jardins's line of reasoning. First, one may be reluctant to adopt his procedure on the whole, that is, to rely upon a direct correspondence between the ratios of lengths of the Line's sections and the 'ratios' of kinds of objects. Whether or not Plato himself regarded such a 'quantitative' correspondence relevant, he left no notice of it. Second, even if the procedure can be accepted in general, the most important premise of Des Jardins's inference, the 'equality' of the class of *eikones* and the 'sum' of the two upper classes of the intelligible objects, looks rather arbitrary. In fact, an equation of the lowest grade of existents (A) with the highest realm of the intelligible $(C + D)$ would hardly conform to Plato's invariably disdainful attitude to the former. Indeed, the lack of body in the Forms is one essential property of the latter making them *superior* to the objects having body.[18] The lack of 'body' (or of depth) in *eikones*, on the other hand, is an accidental property of them that makes them *inferior* to the sensible things endowed with bodies. Since *eikones* and intelligible objects are therefore not bodiless in one and the same respect, it seems incorrect to 'equate' thems as Des Jardins's procedure requires.

4. The 'Specific Ratio' Argument

Suppose one wishes to follow Plato's instructions (509d–510a) for dividing the Line. Since Socrates does not tell Glaucon in what *particular* ratio the Line should be divided in the first place, one may conjecture (as almost all commentators did) that it does not matter. Des Jardins (1976, 491-492) suggests another interpretation of Plato's directions. Perhaps, Plato is assuming not an arbitrary but some *specific* ratio from the very beginning. The *only* specific and self-explanatory[19] ratio that can be involved in a simple division of a line into two unequal parts is the mean and extreme ratio.

Now if the ratio in which the Line has to be divided were, indeed, known in advance, this would provide a unique and reliable prescription for someone who might want to *build* the Line *up*, rather than divide it. One would start by $\frac{B}{A} = \phi$. The next step naturally suggested by the mean and extreme ratio would be to add to A and B another section $D = A + B$. In order to have a true *proportion*, one would finally have to duplicate the middle part $C = B$. This being done, the final result $\frac{D}{C} = \frac{C + D}{A + B} = \phi$ would follow *automatically*.[20]

I take Des Jardins to be suggesting (1976, 494) that Plato himself may have proceeded in precisely this way. Could he not have begun with the golden ratio $\frac{B}{A} = \phi$, then have extended his Line to include C and D, and finally have *discovered* the remarkable property of the Line thus built up $\frac{D}{C}=\frac{B}{A}=\frac{C+D}{A+B}$, and have *incorporated* this property in his account? The way he presents the whole situation in 509d–510b is just the reverse, but we know that the 'context of discovery' is not bound to coincide with the 'context of justification'.

It is fairly legitimate to claim that Plato could, in principle, have proceeded this way, as long as one does not insist that Plato in fact did. But even the abstract possibility of such a reconstruction of the context of Plato's 'discovery' of the remarkable properties alleged by Des Jardins to attach to the Line is seriously afflicted by the fact that no mention of the golden ratio is included in the 'context of justification'. To make the 'specific ratio' argument workable, one has to demonstrate that, at the time when Plato was writing the *Republic* VI, the golden ratio was not only established as a specific mathematical object but was in fact treated as *the Ratio*—as something well-known and not in need of further specification.

The historical evidence relevant to such a conjecture is ambiguous. The golden section was first thoroughly investigated by Euclid in the *Elements*.[21] This section is used in Book IV for the construction of a pentagon (see, *e.g.*, Huntley 1970, ch. 2), which suggests the Pythagorean origin of the corresponding theorems involving ϕ (Heath 1921, 304, 324). Plato thus could have been aware of such 'theorems', and the famous Proclus' commentary seems to support the hypothesis of such an awareness. In this commentary, we are told that Eudoxus 'increased the number of the (propositions) about *the section* which originated with Plato' (as quoted in Heath 1921, 324). If 'the section' here means the golden section, the 'specific ratio' argument may gain some momentum. However, the cited passage from Proclus lends itself easily to very different interpretation (see, *e.g.*, Heath 1921, 325; Sayre 1983, 111-112), and the available historical data are insufficient to narrow down the range of such interpretations.[22]

Nevertheless, even if the golden ratio did not figure so prominently in the pre-Euclidean mathematics as to deserve the name '*the* ratio', the theory of the golden section was definitely emerging at Plato's time, and he could well become familiar with it. If so, it is psychologically plausible to assume that the idea of the golden ratio was present in Plato's mind when he was introducing the unequal division of the Line. He may have been trying it on in his thoughts, in order to figure out how it fits in with his other ideas in the neighborhood (the Sun, the Cave) and elsewhere. He may have finally decided to abandon the golden ratio as being not very important to his purposes. In this case, the Line as we know it, may be a final result of the initial deliberation that somehow included the golden ratio, and of the subsequent decision to exclude it from the written account. Some traces of such a deliberation may still be present in Plato's work, and it is hard to resist the temptation to examine them further.

5. The Epistemological Argument.

The Greek notions of rational and irrational numbers (those expressible by ratio (*logos*) of commensurable magnitudes and those 'without ratio' (*alogoi*) that cannot be so expressed) bear an affinity to the wider notions of rationality and its opposite. To give an account (*logos)* of something is to provide a rational reason for it. The existence of irrational numbers, in this sense, is more than a mathematical fact. It can have certain repercussions for knowledge and practical life. If someone is 'as irrational as the lines so called in geometry', he should not be allowed 'to hold rule in the state' (534d). The geometrical lines can be 'irrational' in only one sense—that of incommensurability with some other lines. A person holding rule in the state should likewise be supposed to be 'commensurable' with it. Irrationality, interpreted as incommensurability, may pose problems for practical life. It may also pose problems for knowledge and understanding. At *Statesman* 257b, the real values of the three types (Statesman, Philosopher, and Sophist) mentioned by Theodorus turn out to relate in a way 'that defies all [Theodorus'] mathematical expressions of proportion;'[23] that is, these relations are somehow 'incommensurable' with his understanding.

One can apply this account of rationality and irrationality to the epistemological context of the Divided Line. Plato's influential idea that knowledge is true judgment (*doxa*) accompanied with an account (*logos*)[24] can be exploited in this context.[25] At 534a, Plato identifies section D with $epist\bar{e}m\bar{e}$, whereas the whole region of the visible $(A+B)$ is now called *doxa*. For an account (*logos*), that is supposed to bridge the gap between $epist\bar{e}m\bar{e}$ and (true) *doxa*, to exist, the latter (namely, *doxa* and $epist\bar{e}m\bar{e}$) should be 'commensurable' with each other. In our notation, $\frac{D}{A+B}=\frac{ar^2}{a+ar}=\frac{r^2}{1+r}$ must be a rational number. Let us call it α. Thus the necessary and sufficient condition for commensurability of D and $A+B$ is defined by

$$\alpha=\frac{r^2}{1+r}\in R, \qquad (1)$$

where R is a class of rational numbers.

It can be clearly seen that all *rational* divisions of the Line satisfy (1). Indeed, if $r\in R$, then $\alpha=\frac{r^2}{1+r}\in R$. This result is trivial and not particularly illuminating. More interesting would be to find out which of the *irrational* divisions of the Line ($r\notin R$), if any, meets the criterion (1). In terms of α, r can be expressed as follows:

$$r=\frac{1+\sqrt{1+4\alpha}}{2\alpha}. \qquad (2)$$

We are now interested in the class of solutions of (2), such that $\alpha\in R$ and $r\notin R$. This class includes no 'simple' irrationals expressible by square roots of non-square numbers. On the other hand, any $\alpha\in R$, such that $1+4\alpha$ is neither square nor a ratio of squares, will give a member of the class sought. For example, $\alpha=\frac{1}{2}$ will generate $r=1+\sqrt{3}$ and the corresponding division of the Line[26] $\{1,\ 1+\sqrt{3},\ 1+\sqrt{3},\ 4+2\sqrt{3}\}$, for which D is indeed commensurable with $A+B$, inasmuch as $\frac{D}{A+B}=2$.

On closer inspection, however, this result does not seem entirely satisfactory. To complete the mathematical imagery of the original epistemological problem (that of knowledge as true belief plus an account), it is not enough that D be commensurable with $(A+B)$. Given the relevant epistemological sense of *logos* as reason or explanation, it seems required that an account (or *logos*), which is supposed to fill in the gap between true belief and knowledge, should itself be rational as well. In other words, this gap symbolized by $D-(A+B)$ should be capable of being filled in with a *rational logos*. Recalling that the general form of the division of the Line is $\{a,\ ar,\ ar,\ ar^2\}$, one can always choose a suitable value of a (which may be irrational, as well as rational—see note 9), so that not only $\alpha = \dfrac{D}{A+B} \in R$, but also:

$$[D-(A+B)] \in R. \qquad (3)$$

Thus, by multiplying all sections of the division in the above example, $\{1,\ 1+\sqrt{3},\ 1+\sqrt{3},\ 4+2\sqrt{3}\}$, *already* satisfying the condition (1), by $\dfrac{1}{2+\sqrt{3}}$, we arrive at the division $\{2-\sqrt{3},\ \sqrt{3}-1,\ \sqrt{3}-1,\ 2\}$, for which $D-(A+B)$, presumably 'standing' for an account (*logos*), is rational.[27]

In other words, it is always possible to meet both conditions (1) and (3), by introducing an appropriate 'scale', or 'dimensional', factor a. But this means that one has to attach importance to the *absolute* lengths of the Line's sections (and, hence, of the Line on the whole), and not only to their ratios. As was already noted, there is no indication in Plato's account of the Line that he was in the least concerned with such absolute values. In the language of modern science, Plato's statement of the problem is *scale-invariant.* One wonders *if* there is a scale-invariant way to meet the condition (3).

There is *only one* such way. We should require that:

$$D-(A+B)=0, \qquad (4)$$

in order to deprive the 'scale factor' a of any significance, and to make the adequacy of the particular division of the Line to the epistemological problem under consideration *independent* of any absolute measure of the Line's length. As we already know, (4) implies $r = \phi$, and we have recovered once again the golden division of the Line, this time—on the epistemological grounds.

Unfortunately, this 'recovery' is not without its own problems. The whole argument is essentially based on the account of knowledge operative in the *Theaetetus*:[28] $epist\bar{e}m\bar{e}$ = true *doxa* + *logos*. Strictly speaking, no such account is found in the proper context of the Line. This is not to say that Plato would regard this account incompatible with what he wrote in the *Republic*.[29] This is only to say that he himself did not attempt to cast his 'justified-true-belief' analysis of knowledge in terms of *eikasia* and/or *pistis*. What he in fact did is something else.

In 525c–533e, Plato makes it clear that *dianoia* falls short of genuine knowledge ($epist\bar{e}m\bar{e}$). Geometry and the 'kindred arts' are only 'dreaming about being, but the clear waking vision of it is impossible for them as long as they leave the assumptions which they employ undisturbed and *cannot give any account of them*' (533c, my emphasis). The requisite account can only be supplied by dialectic. 'The dialectician is one who can "give and receive an account" of what he knows' (Cornford [1932], 1965, 79). That those possessing *dianoia* are yet to come to have genuine knowledge is also manifest in the contrast drawn at 523a, 525b–d, 526e, 529b–e and elsewhere, between the 'vulgar utilitarian' uses of the branches of mathematics and their employment 'for the investigation of the beautiful and the good' (531c).

The analysis of knowledge at 525c–533e thus differs from that of *Theaetetus* and cannot formally be captured by the 'JTB' formula. But even an 'informal' and tentative attempt to import this formula into the context of this part of the *Republic* would not lead to a desired result. Assume, by analogy, that an account (*logos*) should fill in the gap between C (*dianoia*) and D ($epist\bar{e}m\bar{e}$), rather than between $A + B$ and D. Given $r = \phi$ or, in general, $r \notin R$, there is no possibility to accomplish this task in a 'rational' way.[30] This can also be shown in a more general fashion, regardless of whether it is possible to apply the JTB formula to the relation of *dianoia* and $epist\bar{e}m\bar{e}$. A reasonable necessary condition of ‘convertibility’ of *dianoia* (C) into *epistēmē* (D) would be to require that $\frac{D}{C}$ be rational. But clearly $r = \phi = \frac{D}{C}$ does not meet this requirement, insofar as ϕ is not rational itself.

6. The Aesthetic Argument

The golden section is a hallmark of beauty (see, *e.g.*, Huntley 1970). Since the Line is certainly related, albeit indirectly, to the Sun and the Cave similes, the aesthetic value of the golden division would make the latter more than appropriate in the context where the Beautiful is held to be attendant to the Good (505d, 506e, 531c).[31] Though this argument can probably reinforce the power of other well-established reasons (if any) favoring the golden division of the Line, it is hardly sufficient by itself.[32] The Beautiful does come up several times in the *Republic* VI-VII. But it is by no means a prominent factor in the Sun–Line–Cave. In such circumstances, to make the value of the Beautiful solely responsible for the golden division of the Line would indeed be folly.

7. The ‘Driving Ratio’ Argument

By the late fifth century B.C. (Heath 1921, 91ff.), Greek mathematicians have developed two different methods of representing irrational numbers—the geometrical method, and the one based on series of successive ratios of commensurable magnitudes (the ‘side-and-diagonal’ method). According to the latter method, $\sqrt{2}$, for instance, can be produced by the following series:

$$x_1 = 1,\ \ x_2 = 1,\ \ x_{2n+1} = x_{2n-1} + x_{2n},\ \ x_{2n+2} = x_{2n-1} + x_{2n+1} \quad (n = 1, 2, \ldots);$$
$$\lim_{n\to\infty} \frac{x_{2n+2}}{x_{2n+1}} = \sqrt{2}. \qquad (5)$$

This is, of course, a modern notation. The Greeks did not know the formal notion of a limit. Informally, however, the successive *rational* ratios could be considered as ‘tending’ to approximate a particular irrational number.

It is easy to turn the series (5), standing for the most well-known irrational number, into the *Fibonacci series*:

$$f_1 = 1, \ \ f_2 = 1, \ \ f_{2n+1} = f_{2n-1} + f_{2n}, \ \ f_{2n+2} = f_{2n} + f_{2n+1}, \quad n = 1, 2, \ldots \, . \qquad (6)$$

It is well-known that $\lim_{n\to\infty} \frac{f_{2n+2}}{f_{2n+1}} = \lim_{n\to\infty} \frac{f_{2n+1}}{f_{2n}} = \phi.$ Thus, anyone familiar with the series (5) for $\sqrt{2}$ could have quite naturally stumbled across the golden ratio series (6).

Now consider a series of *triplets* of successive Fibonacci numbers $\{ f_n, \ f_{n+1}, \ f_{n+2} \}, \ n = 1, \ 2, \ \ldots$. One can cook up a *quadruplet* out of each such triplet, by duplicating the middle term:

$$\{ f_n, \ f_{n+1}, \ f_{n+1}, \ f_{n+2} \}, \ n = 1, \ 2, \ \ldots \, . \qquad (7)$$

It can be seen that any division of a line in ratios given by such a quadruplet approximates the golden division of Plato’s Line, as n tends to infinity, that is:

$$\lim_{n\to\infty} \frac{f_{n+1}}{f_n} = \lim_{n\to\infty} \frac{f_{n+2}}{f_{n+1}} = \lim_{n\to\infty} \frac{f_{n+1} + f_{n+2}}{f_n + f_{n+1}} = \phi.$$

For each n, we have a *rational* division of the Line, none exactly satisfying Plato’s requirement. Potentially, however, the series (7) is equivalent to the irrational (namely, the golden) division, in the same sense as that in which the series of rational magnitudes (5) is potentially equivalent to $\sqrt{2}$. To turn potentiality into actuality, one has to replace the ‘static’ picture of a ‘completed’ division with the dynamic pattern of successive approximations, in which any particular division is not completed but 'driven', as it were, to the next step in the series (7):

$$\{1,\ 1,\ 1,\ 2\} \rightarrow \{1,\ 2,\ 2,\ 3\} \rightarrow \ldots \rightarrow \{5,\ 8,\ 8,\ 13\} \rightarrow \{8,\ 13,\ 13,\ 21\} \rightarrow \ldots$$

In a recent paper, Dreher (1990) interpreted Plato's Divided Line in this way. He argued that 'Plato's figure zestfully taps into a main current of 5th and 4th century Greek geometry at the same time as it models his view that any cognitive success achieved by the mind intensifies the passion for further inquiry' (Dreher 1990, 159-160).

The problem with this otherwise very appealing interpretation is that it does not conform to Plato's *epistemology* in the *Republic*. Dreher's interpretation seems to presuppose the *open* and never completed status of knowledge. Roughly speaking, 'absolute truth', or complete knowledge, should be conceived of as an actually *unattainable* limit of the infinite process of successive approximations. Corresponding to this limit state of knowledge is the *exact* golden division of the Line, never achieved in actuality.

This ideal pattern of the infinite cumulative growth of knowledge would perhaps be warmly greeted by some scientists of today. But it is certainly very far from what Plato tells us in *Republic* VI and VII. The process whereby a dialectician apprehends the basic non-hypothetical principle of all knowledge can *actually* be *completed* (see, *e.g.*, 511b8, 511d4, 516b, 532b1). No infinite progress suggested by the 'driving ratio' is required for attaining the *anhypothetos archē*. Dreher's image is misleading as far as its epistemological connotations are concerned. But these seem to have been the main reason for introducing this image in the first place.

Dreher discusses and rebuts another objection against the golden division of the Line raised by Sayre (1983, 304n16): the mean and *extreme* ratio implies the extended equality $\frac{B}{A} = \frac{D}{C} = \frac{C+D}{A+B} = \frac{A+B+C+D}{C+D}$. As Sayre points out,[33] it is not easy to make sense of this extension. Dreher's response is that the infinite series (7) is not afflicted by this problem, for $A+B+C+D=f_n+f_{n+1}+f_{n+1}+f_{n+2}$ naturally transforms, at the next step, into $C+D=f_{(n+1)+1}+f_{(n+1)+2}$.

It seems, however, that Dreher underestimates the seriousness of the problem. One is to make sense *not* of the *numerical* relations of terms of the extended equality, but of the relations of their *referents* in the Line. Can the *whole* of the Line become a term in such relations? For example, if the visible is taken to be the image of the intelligible, how can the whole, including the intelligible, be imaged by the latter? In general, as Des Jardins rightly observed, 'since the whole cannot exclude one of its own parts, it cannot take part in any relation founded on mutual exclusion of the parts' (Des Jardins 1976, 494).

This quandary arises for anyone proposing to divide the Line in the mean and extreme ratio. The very feature that makes the golden cut 'golden'—that the ratio of its parts is equal to that between a part and the whole—becomes an obstacle in the way of reading it into Plato's Divided Line.

8. Conclusion

Any attempt to identify Plato's division of the Line with the golden ratio confronts two main problems: (1) the lack of textual evidence for such an identification, and (2) the lack of unambiguous historical evidence of Plato's acquaintance, at the time when he was writing the *Republic*, with the emerging theory of the golden section. All the arguments considered above are afflicted, though to a different extent, by both (1) and (2). Of these two problems, (2) is more open-ended, whereas (1) has a stronger 'no-go' effect on any proposal to divide Plato's Line in the mean and extreme ratio.[34]

Notes

[1]From the equation: $\frac{B}{A} = \frac{C + D}{A + B} = \frac{D}{C}$, it immediately follows that $B = C$. Plato was possibly (but not necessarily) aware of this interesting consequence of Socrates' instructions for dividing the Line. See note 17.

[2]All quotations from the *Republic* are given in P. Shorey's translation.

[3]Plato does not identify the objects of *dianoia* with any particular class of beings. This gave rise to the 'problem of *dianoia*' and extensive debates about the nature of its objects (see, *e.g.*, Boyle 1954; Klein 1965, 155f; Wu 1969; Tanner 1970; Smith 1981). I give *C* the tentative label 'hypothetical objects of mathematics' in full view of the provisional character of any such designation.

[4]Klein (1965, 124) explains this interchange by drawing a parallel between *eikasia* and *dianoia*. The latter is none other than 'dianoetic eikasia' (*ibid*., 199f), a faculty similar to *eikasia* in that it deals with objects of the highest realm of being (Forms) as 'imaged' by 'hypothetical objects of mathematics'. This makes the relation: 'dialectical *noēsis* (or *epistēmē*) is to *pistis* as natural and technical *dianoia* is to *eikasia*' (*ibid*., 124)—intimated by Socrates at 534a—look less puzzling.

[5]Most commentators, in fact, maintain that the image-imaged relation applies all throughout the Line. See, *e.g.,* Tanner (1970), Boyle (1973), Dreher (1990).

[6]One exception being Des Jardins (1976). I shall examine his reasons later on.

[7]Whether one should be pleased with this freedom of interpretation is, of course, a different question. Dreher, for example, argues that the division of the Line is "open-ended in a delightful way" (1990, 159). I am of a different opinion. The reasons are given below.

[8]See Sayre 1983, 303-304n16. I use here a notation slightly different from Sayre's.

[9]Notice that the irrationality of a does not matter, for, as was mentioned, the absolute length of the Line and of its sections is insignificant.

[10]Brumbaugh (1954, 270) seems to be the first to have suggested the possibility of the golden division of Plato's Line (see Dreher 1990, 170). Brumbaugh, however, did not pursue this idea any further.

[11]I noted earlier that, according to Rose (1964), Plato's Divided Line should be reinterpreted as a 'divided rectangle'.

[12]Of course, one could still use such imagery. Euclid, for example, extensively employed line segments for representing rational ratios of numbers in the *Elements* VII–IX.

[13]Eudoxus' theory of proportions uses whole numbers to define equality of ratios in a manner applying to rational and irrational ratios alike. Sayre argues (1983, 103ff) that Eudoxus' theory may have played a major role in Plato's late ontological development.

[14]My thanks to an anonymous referee who pointed out this mathematical flaw in Des Jardins's reasoning.

[15]Given $B > 0$, $A = C + D$ implies $A + B > C + D$; hence, given $\frac{A}{B} = \frac{C}{D} = \frac{A+B}{C+D}$, $A > B$ and $C > D$. This runs counter to the view, adopted by the majority of commentators, that the section standing for the intelligible must be longer than that standing for the visible. However, as I noted above, Plato does not authorize either alternative. Therefore, one should welcome 'minority reports'.

[16]See, however, note 3.

[17]As mentioned (see note 1), $B = C$ is a straightforward consequence of Plato's instructions for dividing the Line. To make Des Jardins's (implicit) claim of 'independent corroboration' legitimate, one has to be sure that Plato himself was aware of $B = C$. No direct evidence for such an awareness is available. However, 534a8–10,

where Socrates 'flip-flops' B and C, indirectly suggests it. See Sayre 1983, 303n13; Desjardins 1990, 175-176; and note 4, in this regard.

[18]Such as, for example, heavenly bodies (*Republic* 530b).

[19]In the sense that it does not require any concrete quantitative 'input' information. As Des Jardins puts it, "If for any reason one has to divide the line merely from Socrates' first sentence, one could turn to the mean and extreme proportion, since it is the only proportion of these three terms' (*ibid*., 492). 'Three terms' here are two sections of the Line and their sum.

[20]Cf.: Des Jardins 1976, 494.

[21]II.11; IV.11-14; VI.30; XIII.1-6, 8-9.

[22]It should be noted that Euclid always refers to the mean and extreme ratio explicitly and never treats it as *the* ratio.

[23]Translation by J. B. Skemp.

[24]This idea constitutes the third definition of knowledge in the *Theaetetus*. In this regard see, for example, Polansky's recent (1992, 209ff). The 'justified true belief' analysis of knowledge is also found, with minor variations, at *Meno* 97d–98e, *Symposium* 202a, and elsewhere.

[25]Desjardins (1990, 177ff) has recently suggested one way of doing it. The approach developed below is different from Desjardins's.

[26]For simplicity, we take $a = 1$.

[27]In fact, $D-(A+B)=1$.

[28]As well as in the *Meno* and the *Symposium*. See note 24.

[29]At 531e, for example, Socrates observes that '[m]en who could not render and exact an account of opinions in discussion would never know anything of the things we say must be known'.

[30]That is, to make the difference $D-C$ rational.

[31]Robinson (1953, 181ff) would disagree. He maintained that the Line is not related to the adjacent images of the Sun and the Cave. Therefore, he argued, there is no room for the idea of the Good in the Line.

[32]This suggests a more general question regarding the joint power of different arguments for the golden division. Perhaps none of them is really strong. But, one might insist, when taken together, they produce a 'consilience' or 'cumulative' effect. One should be very careful with this. Each argument is formulated in the framework of a particular interpretation, and these do not always conform to each other. For example, the epistemological argument and Des Jardins's 'metaphysical argument' cannot be jointly employed, for they are based on incompatible assumptions. The epistemological argument assumes $D = A + B$, whereas Des Jardins holds that $A = C + D$.

[33]See also Des Jardins 1976, 494.

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